J. Fluid Mech. (1973), *vol.* 58, *part* 4, *pp.* 709–726
Printed in Great Britain

On the existence of critical levels, with applications to hydromagnetic waves

By JAMES F. McKENZIE
European Space Research Institute, Frascati, Italy

(Received 5 June 1972 and in revised form 12 January 1973)

It is proved that a critical level, at which a wave packet is neither reflected nor transmitted, can exist only if the wave normal curve, which is formed by taking the cross-section through the wave normal surface in the plane of propagation, possesses an asymptote which is parallel to the direction of variation of the properties of the medium through which the wave packet moves. This condition, when applied to various types of hydromagnetic waves (such as hydromagnetic waves of the inertial or gravity type, or slow magnetoacoustic waves), shows that critical levels for such waves can exist only if the direction of spatial variations of the medium is perpendicular to the ambient magnetic field. Provided that the angle between the gravitational acceleration, or the rotation axis, and the magnetic field is not zero, hydromagnetic critical levels, characteristic of the gravity or inertial type, act like 'valves' in the sense that the wave packet can pierce the critical level from one side and is captured from the other side. It is also pointed out that critical-level behaviour is to some extent a consequence of the WKBJ approximation since the other limit, namely when the waves feel an almost discontinuous behaviour in the properties of the medium, gives markedly different results, particularly in the presence of streaming, which can give rise to the phenomenon of wave amplification.

1. Introduction

The concept of a critical level, at which a wave packet is neither reflected nor transmitted, emerged from a study, using the WKBJ approximation, of the propagation of internal gravity waves in a shear flow (Bretherton 1966). Subsequently, a more refined analysis (Booker & Bretherton 1967) showed that a gravity wave group can be transmitted through a critical level but is heavily attenuated for Richardson numbers of the order of unity or greater. More recently it has been pointed out (Acheson 1972; Rudraiah & Venkatachalappa 1972) that hydromagnetic wave groups in stratified rotating fluids can also exhibit 'critical-level' behaviour.

One of the purposes of this paper is to point out that the existence of a critical level for any type of wave propagation in a stratified medium depends on the wave normal surface reaching to infinity in a certain manner. In fact, in the next section we prove that *if the wave normal curve, formed by taking the cross-section through the wave normal surface in the plane of propagation, possesses an asymptote, a critical level can exist provided that the properties of the medium vary in a direction* *parallel to the asymptote*. This condition provides a more unified view on the possible existence of critical levels than has been recognized hitherto.

In §§ 3–5 we discuss the propagation properties of various types of hydromagnetic waves (viz. hydromagnetic–gravity waves, magnetoacoustic waves and hydromagnetic–inertial waves) in terms of the geometry of their corresponding wave normal surfaces. We find, on applying the condition for the existence of a critical level, that such waves exhibit critical levels provided that the *direction of spatial variations of the medium is perpendicular to the direction of the magnetic field*. In addition, hydromagnetic waves approach their critical levels from one side only. The reason for this behaviour, which contrasts with that of gravity waves in a shear flow, is that the asymptote of the wave normal curve is approached from only one side. In the case of hydromagnetic–gravity waves and hydromagnetic–inertial waves the ray can pass through the critical level from one side but not from the other. In this sense *such critical levels behave like 'valves'*. This effect was first discovered in the context of hydromagnetic–inertial wave propagation by Acheson (1972), who also emphasized the important proviso above. Diagrams illustrating typical ray trajectories (of which there is a rich variety) for the various types of waves are obtained by using the geometry of the wave normal surfaces.

In the concluding section we point out that critical-level behaviour must somehow be a symptom of the WKBJ approximation (which is incapable of accounting for partial reflexions) since the other asymptotic limit, in which the waves experience an almost discontinuous jump in the properties of the medium, yields very different results. An exact analysis would probably yield a curve (or curves), in the parameter plane of the vertical wavelength normalized by the scale of variation of the medium versus some other parameter characterizing the medium (e.g. the Richardson number in the case of a shear flow), which separates these two regimes (Jones 1968; McKenzie 1972).

2. The condition for a critical level in wave propagation

For simplicity we confine our discussion to the propagation of waves, in the WKB approximation, in a medium whose properties vary with only one Cartesian co-ordinate, z say, and we orient the horizontal co-ordinate, x say, along the horizontal direction of propagation. The equations of the system in general yield a dispersion equation

$$D(\omega, k_x, k_z, z) = 0, \tag{1}$$

which relates the frequency ω to the wavenumbers k_x and k_z at each 'altitude' z. Alternatively, since the properties of the medium vary in the z direction (1) can be regarded as determining k_z at each height, so that (1), solved for k_z, can be written as

$$k_z = k_z(\omega, k_x, z). \tag{2}$$

(Equation (1), solved for k_z, can yield more than one branch, so we fix our attention on one branch (or mode).) The trajectory of a wave packet (a ray) is given by (see, e.g. Lighthill 1965)

$$\frac{dx}{dt} = \frac{\partial\omega}{\partial k_x} = \frac{\partial D/\partial k_x}{\partial D/\partial\omega}, \tag{3a}$$

$$\frac{dz}{dt} = \frac{\partial \omega}{\partial k_z} = \frac{\partial D/\partial k_z}{\partial D/\partial \omega}, \tag{3b}$$

in which ω and k_x are conserved along the path of the ray. Dividing (3*a*) by (3*b*) to eliminate t we obtain

$$\frac{dx}{dz} = \frac{\partial D/\partial k_x}{\partial D/\partial k_z} = -\frac{\partial k_z}{\partial k_x}, \tag{4a}$$

which is the equation for the ray trajectory in the z, x plane, and can be put in the integral form

$$x + \text{constant} = -\int dz \frac{\partial k_z}{\partial k_x}. \tag{4b}$$

We can regard (1) or (2) as defining a curve (the wave normal curve) in the k_z, k_x plane for any fixed values of ω and z. If for some range of values of z the wave normal curve possesses an asymptote such that $k_z \to \infty$ as $k_x \to k_\infty(z)$ a critical level can exist at an altitude z_c given by

$$k_x = k_\infty(z_c). \tag{5}$$

This follows from (4*a*), which shows that dx/dz and all higher derivatives are infinite at $z = z_c$.

For example if, for some range of values of z, the wave normal curve is asymptotic to a line $k_x = k_\infty(z)$ such that

$$k_z \sim 1/|k_\infty(z) - k_x|^\beta \quad (\beta > 0), \tag{6}$$

equations (4) show that the ray approaches a critical level ($z \to z_c$) in the fashion

$$x \sim 1/|z_c - z|^\beta. \tag{7}$$

Consider the familiar case of the propagation of gravity waves in a shear flow. The dispersion equation (Bretherton 1966) is

$$k_z^2 = N^2 k_x^2/(\omega - k_x U_x(z))^2 - k_x^2, \tag{8}$$

in which N is the Brunt–Väisälä frequency, $U_x(z)$ is the horizontal wind speed, and we restrict attention to the propagation in the z, x plane of a wave packet with dominant frequency ω and horizontal wavenumber k_x, both of which are conserved along the ray trajectory. The wave normal curve (given by (8)) possesses an asymptote of the form given by (5) in which we now have

$$k_\infty(z) = \omega/U_x(z) \quad (\beta = 1). \tag{9}$$

Thus the ray approaches a critical level at $z = z_c$ in the fashion

$$x \sim \begin{cases} 1/(z_c - z) & \text{for} \quad U(z) < \omega/k_x \quad (k_z < 0, \text{ upward propagation}), \\ 1/(z - z_c) & \text{for} \quad U(z) > \omega/k_x \quad (k_z < 0, \text{ downward propagation}), \end{cases}$$

in which z_c is given by

$$k_x = k_\infty(z_c) = \omega/U_x(z_c). \tag{10}$$

We have assumed that the wind speed increases smoothly with height. Equation (10) states that the critical height occurs where the horizontal phase speed matches the wind speed. The reason why a gravity wave can be captured on either side of z_c is that the asymptote of the wave normal curve is approached from both sides.

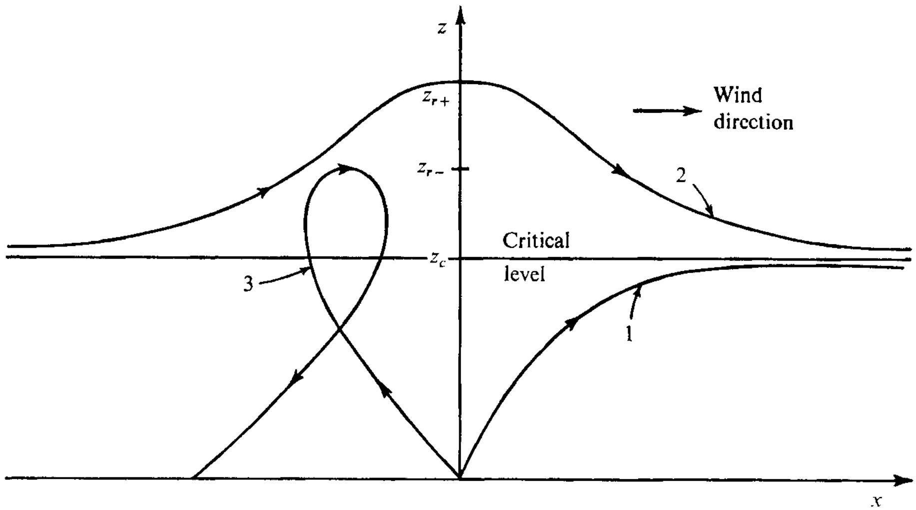

FIGURE 1. Gravity wave ray trajectories in a horizontal wind with speed U increasing with altitude. The rays labelled 1 and 2 are generated below and above the critical $z = z_c$ at which they are captured; $\omega/k_x = U(z_c)$. The ray labelled 3 propagates against the wind and is reflected at $z = z_r$; $|(N \pm w)/k_x| = U(z_{r\pm})$.

By noting that the ray is normal to wave normal curve we can readily follow the path of the ray, along which ω and k_x are conserved, by constructing the shapes of the wave normal curve at successive values of z for any assumed variation of the properties of the medium with z. This geometrical construction (Lighthill 1967) has been used to construct the ray trajectories shown in figures 1, 3, 4, 6 and 9. We note that a wave packet propagating against the wind ($\omega/k_x < 0$) follows a looping trajectory, being reflected at a level where its frequency, as measured in the rest frame ($\omega' = \omega - Uk_x$), matches the Brunt–Väisälä frequency N.

3. Hydromagnetic–gravity waves

In this section we consider the propagation properties of hydromagnetic–gravity waves in a highly conducting, incompressible fluid (Lighthill 1967; Hide 1969). One mode of motion corresponds to the component of the vorticity parallel to the magnetic field being propagated at the Alfvén speed along the field lines. The other mode of motion, which involves an interaction between the magnetic and buoyancy forces, is governed by the dispersion equations

$$\omega^2 = N^2(\hat{\mathbf{g}} \times \mathbf{k})^2/k^2 + (\mathbf{b}\,.\,\mathbf{k})^2, \tag{11}$$

in which $\hat{\mathbf{g}}$ is the unit vector along $\mathbf{g}$; N, the Brunt–Väisälä frequency, and $\mathbf{b}$, the Alfven velocity, are given by

$$N^2 = \rho_0^{-1}\mathbf{g}\,.\,\nabla\rho_0, \quad \mathbf{b} = \mathbf{B}_0/(\mu_0\rho_0)^{\frac{1}{2}}, \tag{12}$$

where ρ_0 is the equilibrium density and $\mathbf{B}_0$ is the ambient field strength. When N and $\mathbf{b}$ are slowly varying functions of position (11) can be regarded as the local dispersion equation.

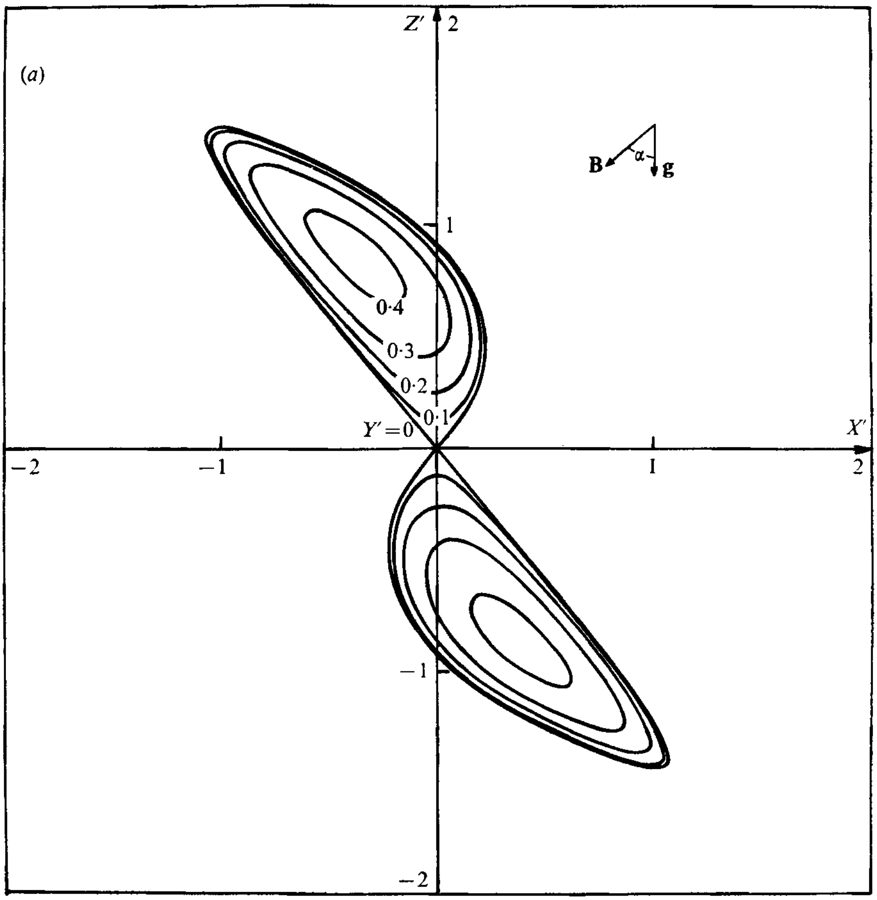

FIGURE 2. For legend see p. 715.

If the normalized variables w and $\mathbf{R}'$ given by

$$w = \omega/N, \quad \mathbf{R}' = (X', Y', Z') = \mathbf{k}/(N/b)$$

are introduced (11) may be written as

$$w^2 = \frac{X'^2+Y'^2}{X'^2+Y'^2+Z'^2} + (X'\sin\alpha + Z'\cos\alpha)^2, \tag{13}$$

in which we have taken the Z', X' plane to contain $\mathbf{g}$ and $\mathbf{B}_0$, and α is the angle between $\mathbf{B}_0$ and $\mathbf{g}$.

At a fixed frequency cross-sections of the wave normal surface taken through planes $Y' =$ constant are as shown in figures 2(*a*), (*b*) and (*c*).

When $w < \cos\alpha$ the Alfvén wave planes $X'\cos\alpha + Z'\cos\alpha = \pm w$ cut the gravity wave cone in ellipses and the resulting wave normal surface is closed, as is shown in figure 2(*a*). The wavenumber Y' is limited to the range

$$|Y'\cos\alpha| < 1-(1-\omega^2)^{\frac{1}{2}}. \tag{14}$$

When $1 > w > \cos\alpha$ the Alfvén wave planes cut the gravity wave cones in hyperbolas and the cross-sections of the wave normal surface have the shapes

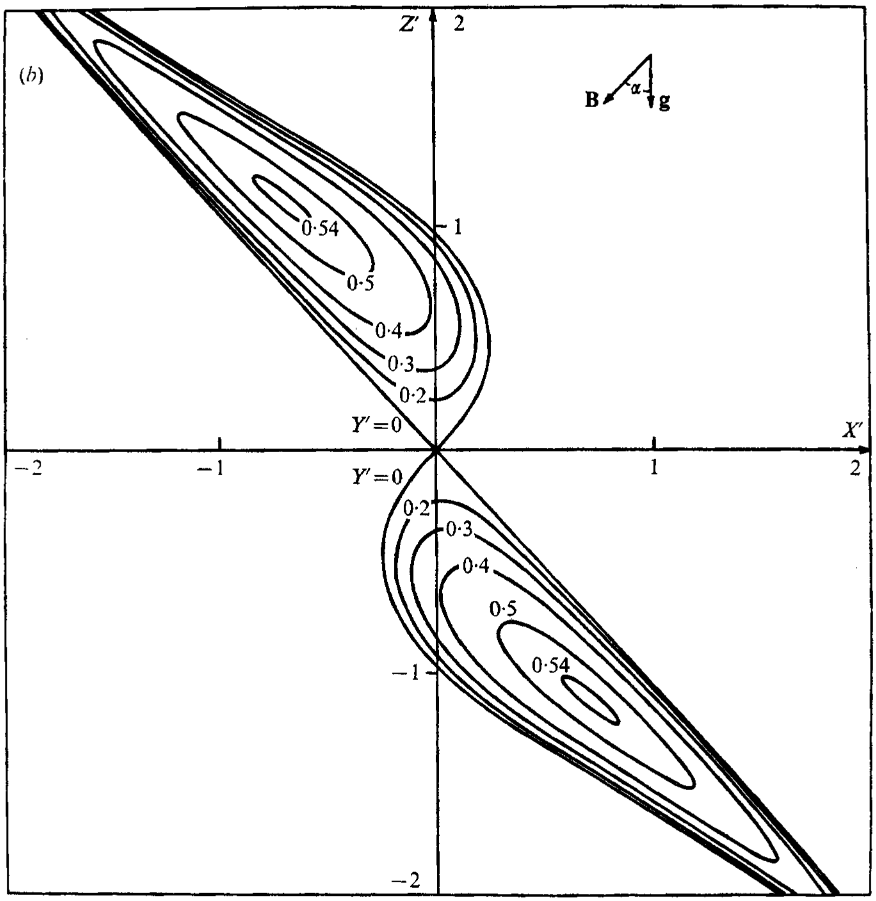

FIGURE 2. For legend see facing page.

shown in figure 2(*b*). In the plane $Y' = 0$ the cross-section is asymptotic to the lines

$$Z'\cos\alpha + X'\sin\alpha = \pm(w^2 - \cos^2\alpha)^{\frac{1}{2}}. \tag{15}$$

The wavenumber Y' is again restricted by relation (14).

Figure 2(*c*) illustrates the wave normal surface when $w > 1$. For $Y' \gg 1$ the surface is asymptotic to the planes

$$X'\sin\alpha + Z'\cos\alpha = \pm(w^2 - 1)^{\frac{1}{2}}, \tag{16}$$

while for $Y' \ll 1, Z' \gg 1$ and $X' \gg 1$ the surface is asymptotic to the plane defined by (15). Therefore the surface can be visualized as a plane, given by (16), with asymmetric bumps either side of a line which lies in the surface and whose equation is

$$X'\sin\alpha = (w^2 - 1)^{\frac{1}{2}}. \tag{17}$$

As has been pointed out by Lighthill (1967), figure 2(*c*) is important because it indicates that in the presence of a magnetic field a wave which started out being predominantly of the gravity-wave type can be transformed to an Alfvén wave and therefore avoid being reflected at a height where the Brunt–Väisälä frequency equals the wave frequency. (This property may be used as a possible mechanism to explain the existence of Alfvén waves in the solar wind.)

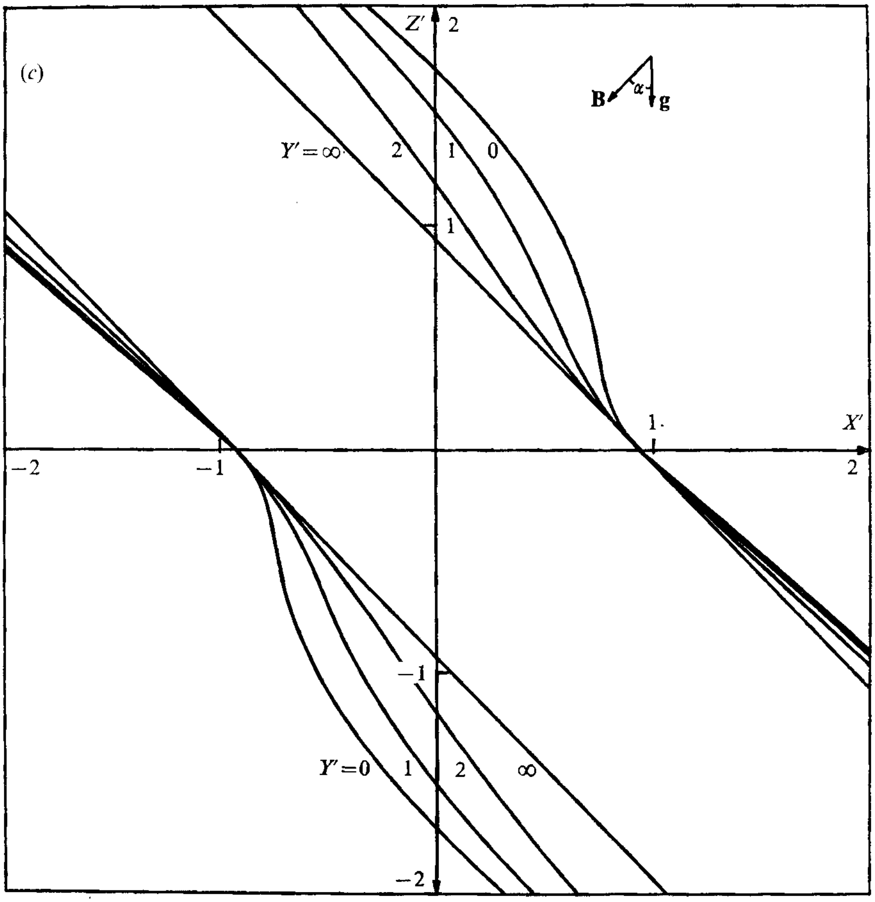

FIGURE 2. Cross-sections of the wave normal surface for hydromagnetic–gravity waves at a fixed frequency for various values of the normalized wavenumber Y'. $\alpha = 45°$ (*a*) $w < \cos\alpha$ ($w = 0{\cdot}65$). (*b*) $1 > w > \cos\alpha$ ($w = 0{\cdot}866$). (*c*) $w > 1$ ($w = 1{\cdot}2$).

More important for our present purpose is that since the asymptotes of the wave normal surface (figures 2*b*, *c*) are perpendicular to the magnetic field direction it is clear that a critical level can only exist if the direction of spatial variations (in either magnetic field or density) is perpendicular to the magnetic field. Consider the case where **B** is parallel to the x axis and **g** lies in the z, x plane, making an angle α with the x axis, and spatial variations in **B** are in the z direction. From (11) we find that the wave normal curve in k_z, k_x plane has an asymptote given by

$$k_z \simeq \frac{-2\sin\alpha\cos\alpha N^2 k_x}{b^2[\{(\omega^2 - N^2\cos^2\alpha)/b^2\} - k_x^2]} \sim \frac{1}{(k_\infty - k_x)(k_\infty + k_x)}. \tag{18}$$

Since the asymptotes for hydromagnetic waves are symmetrically placed with respect to the magnetic field we henceforth limit our discussion to $k_x > 0$. Taking k_x 0 > and comparing (18) with (6) we have that

$$k_\infty = (\omega^2 - N^2(z)\cos^2\alpha)^{\frac{1}{2}}/b(z) \quad (\beta = 1), \tag{19}$$

in the case of coupled hydromagnetic–gravity waves.

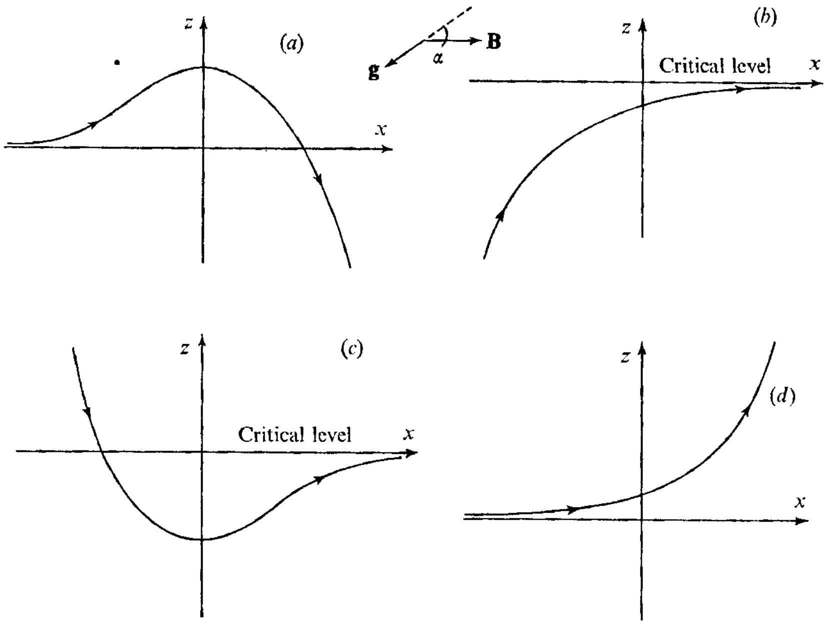

FIGURE 3. Ray trajectories for hydromagnetic–gravity waves in the frequency range $N > \omega > N\cos\alpha$ ($\omega/k_x > 0$) propagating in the plane containing **g** and **B**. (*a*) b decreasing with z, $k_z > 0$. (*b*) b increasing with z, $k_z < 0$. (*c*) b decreasing with z, $k_z > 0$. (*d*) b decreasing with z, $k_z < 0$. Cases (*b*) and (*c*) exhibit critical levels.

Thus we see that at a height $z = z_c$ where the Alfvén speed takes the value given by

$$k_x = (\omega^2 - N^2(z_c)\cos^2\alpha)^{\frac{1}{2}}/b(z_c) \tag{20}$$

a wave packet approaches the x axis (the direction of magnetic field) in the fashion

$$x \sim 1/|z_c - z|.$$

In the special cases $\alpha = \frac{1}{2}\pi$ or 0 (i.e. **g** perpendicular or parallel to **B**) the asymptote is of the form

$$k_z \sim 1/|k_\infty - k_x|^{\frac{1}{2}},$$

so that the ray approaches a critical level in the manner

$$x \sim 1/|z_c - z|^{\frac{1}{2}},$$

where z_c is given by

$$\omega/k_x = b(z_c), \quad \alpha = \tfrac{1}{2}\pi,$$

or

$$(\omega/k_x)(1 - N^2(z_c)/\omega^2)^{\frac{1}{2}} = b(z_c) \quad (\alpha = 0).$$

Typical ray trajectories can be readily determined by using the geometry of the wave normal surfaces. We have done this to construct figures 3 and 4, which show typical trajectories for the case of N constant and b either increasing or decreasing smoothly with z. Figure 3 applies to the case when $N\cos\alpha < \omega < 1$ and the ray propagates in the plane containing **g** and **B**. Cases (*a*) and (*b*) exhibit critical levels. We note that if $k_y \neq 0$, that is if the wavenumber vector has a component perpendicular to the plane containing **g** and **B**, critical levels could not exist in this frequency range since the wave normal curves (see figure 2*b*) do

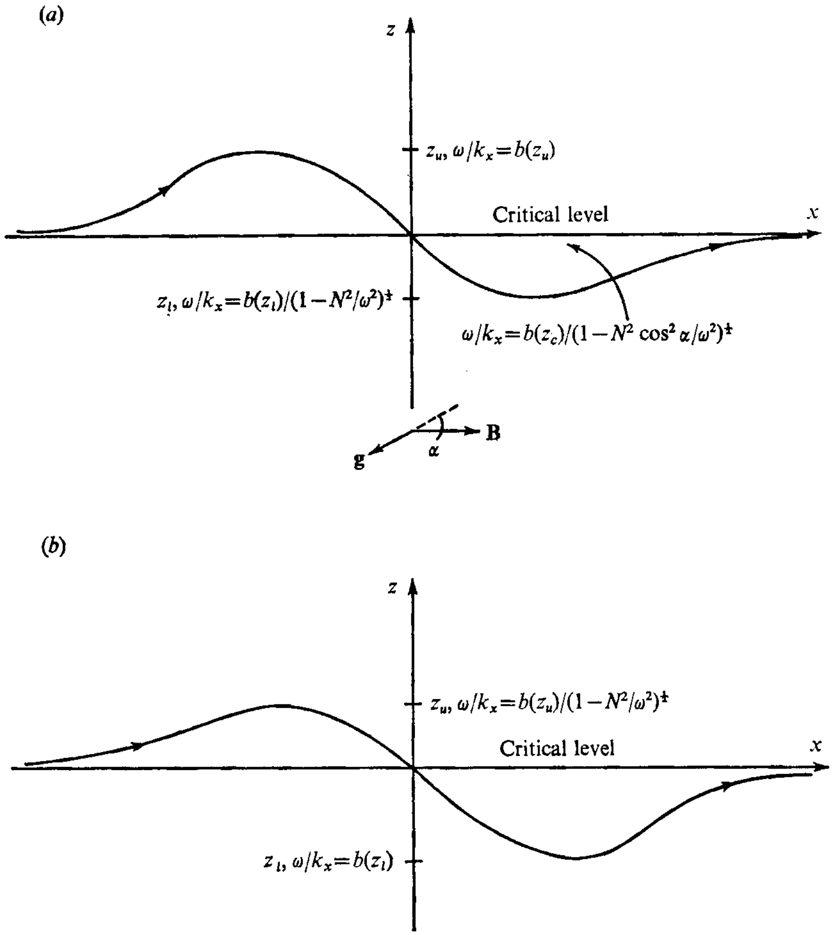

FIGURE 4. Ray trajectories for hydromagnetic–gravity waves, at frequencies $\omega > N$ ($w/k_x > 0$), propagating in the plane containing **g** and **B**. (*a*) b increasing with z. (*b*) b decreasing with z. In both cases the ray approaches the critical level from beneath. If the wave-number vector has a component out of the plane containing **B** and **g** the ray trajectory in the z, x plane is similar to the ray trajectory for $k_y = 0$ and lies between it and the x axis. These trajectories illustrate the 'valve effect'.

not possess asymptotes. The ray trajectories shown in figure 4 apply for frequencies greater than the Brunt–Väisälä frequency. The critical levels are approached from below, $z < z_c$. If $k_y \neq 0$ the corresponding ray trajectory in the z, x plane would be similar to the ray trajectory for $k_y = 0$ and would lie between it and the critical level. We also note that these trajectories illustrate that the critical level for such waves acts like a 'valve' in the sense that the ray can pass through the critical level from above but not from below.

If **g** is either parallel ($\alpha = 0$) or perpendicular ($\alpha = \frac{1}{2}\pi$) to the magnetic field the valve effect disappears and the ray trajectories are of the same type as those for slow magnetoacoustic waves (see figure 6). The disappearance of the valve effect when $\alpha = 0$ or $\frac{1}{2}\pi$ is closely linked with the symmetry of the wave normal curves about the magnetic-field direction.

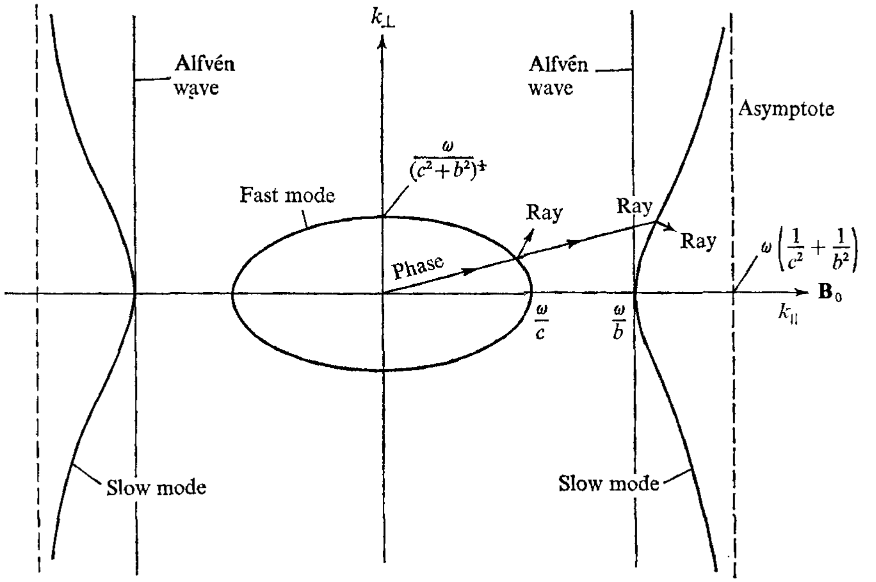

FIGURE 5. Cross-section of the wave normal surface for magnetoacoustic and Alfvén waves (sketched for $c > b$).

4. Magnetoacoustic waves

The theory of the propagation of magnetoacoustic waves in a perfectly conducting, inviscid, compressible medium has been extensively developed (see Lighthill 1960, and many others). The dispersion equation for magnetoacoustic waves is given by

$$k_y^2+k_z^2 = \frac{(\omega^2-c^2k_x^2)(\omega^2-b^2k_x^2)}{(c^2+b^2)\,\omega^2-c^2b^2k_x^2}, \tag{21}$$

in which c is the sound speed and it is assumed that the magnetic field is parallel to the x axis. Figure 5 shows the shape of the wave normal curves (the wave normal surface is obtained by rotation of the wave normal curve around the k_x axis). We see immediately from (21) and figure 5 that the wave normal curve appropriate to the slow magnetoacoustic wave is asymptotic to the lines

$$k_x = \pm\,\omega(1/c^2+1/b^2)^{\frac{1}{2}} = \pm\,k_\infty. \tag{22}$$

Thus, if the properties of the medium (such as the sound or Alfvén speeds) vary in a direction z which is perpendicular to the magnetic field, a critical level can exist for a ray of given ω and k_x at $z = z_c$, where z_c is given by

$$k_x = \omega(1/c^2(z_c)+1/b^2(z_c))^{\frac{1}{2}}, \tag{23}$$

and the ray approaches the critical level in the manner

$$x \sim 1/|z_c-z|^{\frac{1}{2}}.$$

A sketch of the ray trajectory is shown in figure 6 for the case of either b or c monotonically increasing or decreasing with z.

(a)

(b)

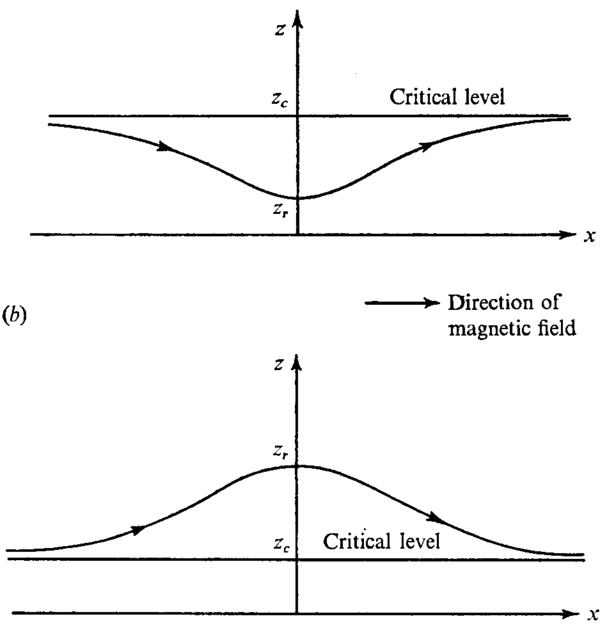

FIGURE 6. Ray trajectories for the slow magnetoacoustic mode exhibiting critical-level behaviour. (*a*) b (or c) increasing with z. (*b*) b (or c) decreasing with z. z_c is given by

$$\omega/k_z = (1/c^2(z_c) + 1/b^2(z_c))^{-\frac{1}{2}};$$

z_r is given by $\omega/k_z = \min(c(z_r), b(z_r))$.

5. Hydromagnetic–inertial waves

The dispersion relation for plane hydromagnetic waves in a highly conducting, incompressible fluid rotating with angular frequency $\mathbf{\Omega}$ is (see Lehnert 1954 or Hide 1969)

$$[\omega^2 - (\mathbf{b}.\mathbf{k})^2]^2 k^2 = (2\mathbf{\Omega}.\mathbf{k}\omega)^2. \tag{24a}$$

Equation (24*a*) may also be written as

$$w = (\mathbf{R}'.\hat{\mathbf{n}})^2 + \frac{1}{2}\frac{Z'^2}{R'^2}\left|1 \pm \left(1 + 4\frac{(\mathbf{R}'.\hat{\mathbf{n}})^2}{Z'^2}R'^2\right)^{\frac{1}{2}}\right|, \tag{24b}$$

in which

$$w = \omega/2\Omega, \quad \mathbf{R}' = (X', Y', Z') = \mathbf{k}/(2\Omega/b), \quad \hat{\mathbf{n}} = \mathbf{b}/b = (\sin\alpha, 0, \cos\alpha),$$

where α is the angle between the magnetic-field direction and the X' axis and we have taken $\mathbf{\Omega}$ to be parallel to the Z' axis.

Equation (24*b*) is a more convenient form for exploring the geometry of the wave normal surfaces. There are two distinct surfaces corresponding to which there are two modes of motion which may be classified as the 'fast' and 'slow' modes according as the phase speed of the mode is faster or slower than the Alfvén speed.

Cross-sections of the two wave normal surfaces taken through planes $Y' =$ constant are shown in figures 7 (*a*), (*b*) and (*c*) for three separate frequencies. The cross-sections of the wave normal surface corresponding to the fast and slow modes are drawn as broken and full lines respectively. When $w < \sin\alpha$ the Alfvén wave

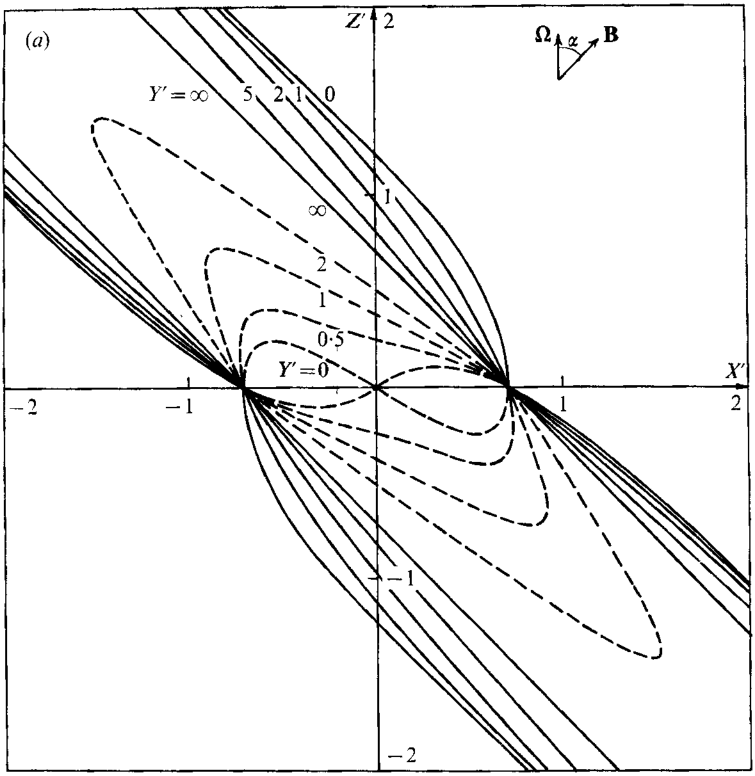

FIGURE 7. For legend see facing page.

planes $\mathbf{R}'.\hat{\mathbf{n}} = \pm w$, cut the inertial wave cone in hyperbolas and the resulting wave normal surface for the fast mode is as shown in figure 7 (*a*). When

$$1 > w > \sin\alpha$$

the Alfvén wave planes cut the inertial wave cone in ellipses and the 'fast' wave normal surface is of the type shown in figure 7 (*b*). In the case $w > 1$ the 'fast' wave normal surface is as shown in figure 7 (*c*). This sequence of shapes for the 'fast' wave normal surface indicates that the inertial mode can be transformed into an Alfvén mode and therefore avoid being reflected at any level where the rotation frequency drops to half the wave frequency.

Both wave normal surfaces are folded along the lines $X' = \pm w$. The basic shape of the wave normal surface associated with the slow mode is unchanged by varying the frequency. The slow mode is predominantly an Alfvén wave modified by the action of the Coriolis force; the effect of the latter becomes particularly pronounced at very low frequencies ($w \ll 1$) and results in wave propagation at speeds very much slower than the Alfvén speed except for propagation corresponding to the region of the folds. If $(\mathbf{b}.\mathbf{k})^2 \ll (\mathbf{\Omega}.\mathbf{k}/k)^2$ the propagation of the fast mode is described by the inertial wave cone

$$\omega^2 = (2\mathbf{\Omega}.\mathbf{k})^2/k^2, \tag{25a}$$

while the slow wave propagates according to the equation

$$\omega^2 \approx (\mathbf{b}.\mathbf{k})^4 k^2/(2\mathbf{\Omega}.\mathbf{k})^2. \tag{25b}$$

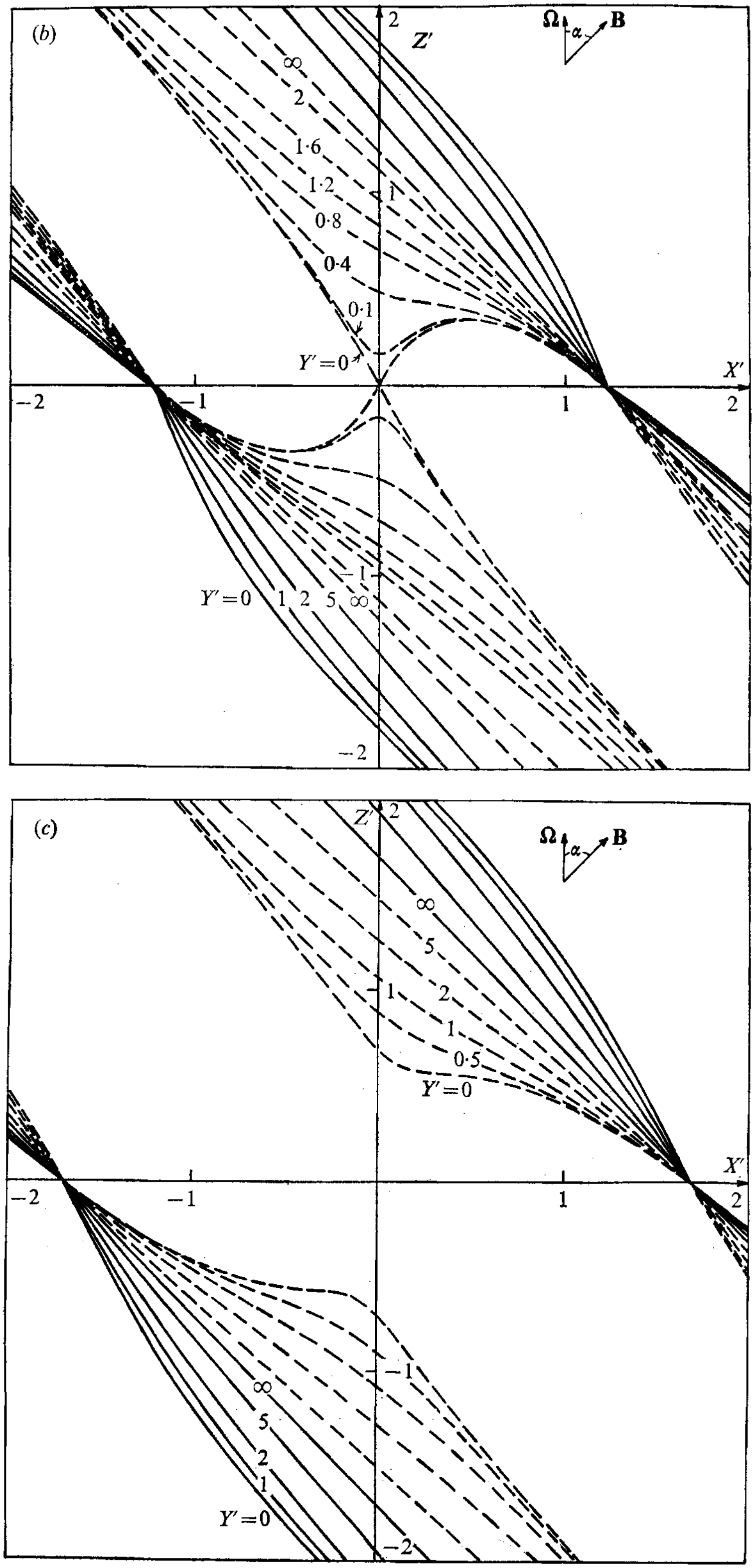

FIGURE 7. Cross-sections of the wave normal surfaces for hydromagnetic–inertial waves, through planes $Y' =$ constant. ———, slow mode; – – –, fast mode. (*a*) $w < \sin\alpha\,(w = 0{\cdot}5)$. (*b*) $1 > w > \sin\alpha\,(w = 0{\cdot}866)$. (*c*) $w > 1\ (w = 1{\cdot}2)$. $\alpha = 45°$.

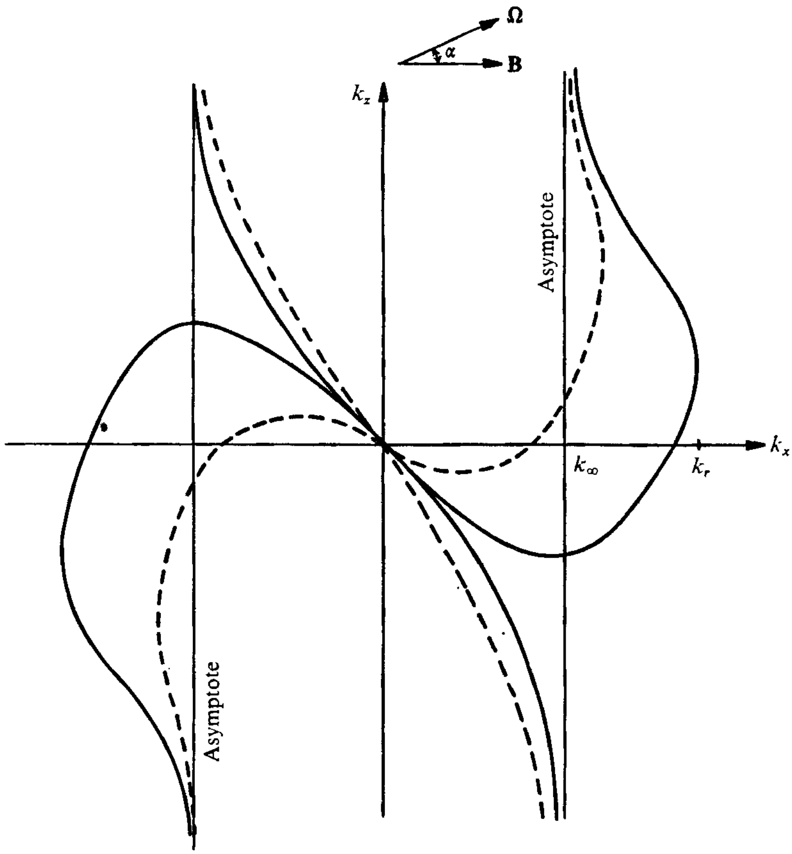

FIGURE 8. Sketch of the cross-sections of the wave normal surface for the slow mode in the approximation $(\mathbf{b}.\mathbf{k})^2 \ll (\mathbf{\Omega}.\mathbf{k}/k)^2$ (equation (25*b*)). ———, $k_y = 0$; – – –, $k_y \neq 0$.
$k_\infty = (\omega^2\Omega \sin\alpha)^{\frac{1}{3}}/b, k_r = (\omega^2\Omega)^{\frac{1}{3}}/b$.

This mode is of particular importance in hydromagnetic oscillations of the earth's core (Hide 1966, Hide & Stewartson 1972). Because this form accentuates certain features that are not too obvious in figure 7 we have sketched, in figure 8, the wave normal curves in the k_z, k_x plane for $k_y = 0$ and $k_y \neq 0$.

Figures 7 and 8 show immediately that critical levels can occur only if the medium varies in a direction z perpendicular to the magnetic field. Taking $\mathbf{B}$ parallel to the x axis, $\mathbf{\Omega} = \Omega(\cos\alpha, 0, \sin\alpha)$ and restricting attention to wave propagation in the z, x plane we find that the wave normal surfaces (given by equations (24)) possess asymptotes of the form

$$k_z \simeq \frac{8\omega^2\Omega^2 \cos\alpha \sin\alpha k_x}{(k^2_{\infty s} - k^2_x)(k^2_{\infty f} - k^2_x)}, \tag{26}$$

in which

$$k^2_{\infty s,f} = (\omega^2 \pm 2\omega\Omega \sin\alpha)/b^2(z). \tag{27}$$

The asymptotes $k_x = \pm k_{\infty s}(\pm k_{\infty f})$ belong to the wave normal surface corresponding to the slow (fast) mode. (The fast mode wave normal surface possesses real asymptotes provided that $\omega > 2\Omega \sin\alpha$.)

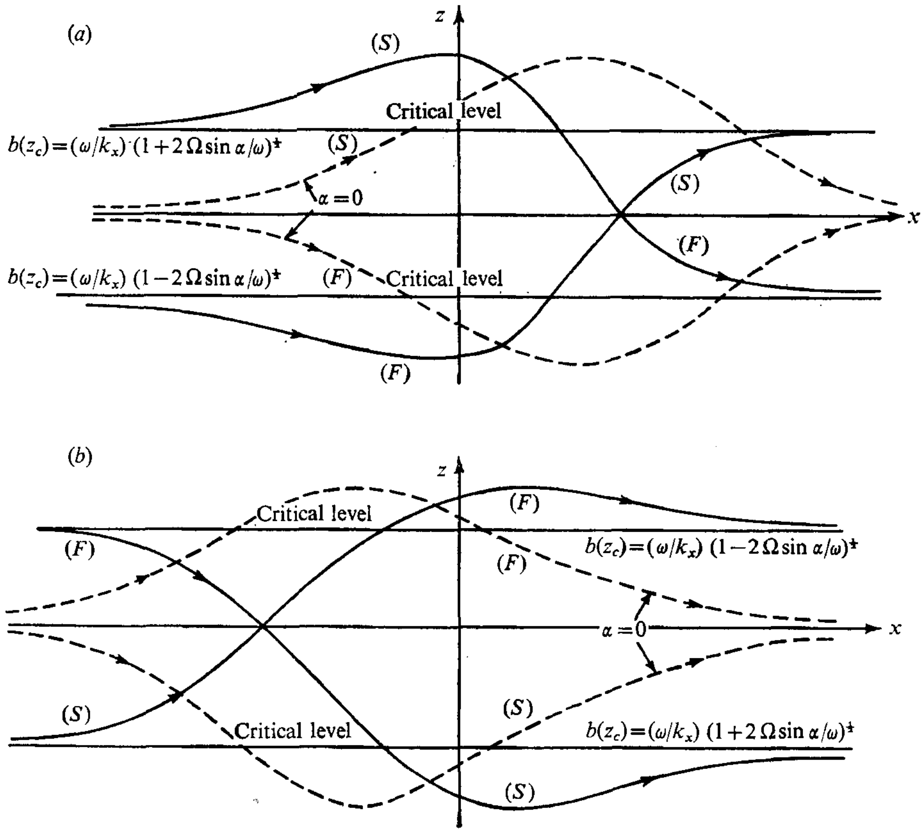

FIGURE 9. Ray trajectories for propagation in the plane containing $\mathbf{\Omega}$ and $\mathbf{B}$ for the slow (S) and fast (F) hydromagnetic–inertial waves. ———, for a general value of α; - - -, $\alpha = 0$. The slow (fast) mode approaches its critical level from below (above). The trajectories (for $\alpha \neq 0$) illustrate the 'valve effect' for hydromagnetic–inertial waves. At the point where the ray trajectories intersect mode conversion takes place so that the slow (fast) mode is transmitted through this point as the fast (slow) mode. (a) b increasing with z. (b) b decreasing with z.

By comparing (27) and (26) with (6) and (5) respectively we can say immediately that critical levels can exist at a height $z = z_c$ given by

$$k_x = \omega(1 \pm 2\Omega \sin\alpha/\omega)^{\frac{1}{2}}/b(z_c), \tag{28a}$$

and the ray trajectory becomes

$$x \sim 1/|z_c - z|;$$

the plus (minus) sign in (28a) refers to the slow (fast) mode.

In the special cases $\alpha = 0$ (rotation axis aligned with magnetic field) and $\alpha = \frac{1}{2}\pi$ (rotation axis perpendicular to the magnetic field) the ray approaches the critical level in the fashion $x \sim |z_c - z|^{-\frac{1}{2}}$, where z_c is now given by

$$k_x = \begin{cases} \omega(1 \pm 2\Omega/\omega)^{\frac{1}{2}}/b(z_c) & \text{for} \quad \alpha = \frac{1}{2}\pi, \\ \omega/b(z_c) & \text{for} \quad \alpha = 0. \end{cases} \tag{28b}$$

Typical ray trajectories for both the slow and fast modes are shown in figure 9. The reflexion points occur at a height $z = z_0$ given by

$$k_x = [\omega(\omega \pm 2\Omega)]^{\frac{1}{2}}/b(z_0). \tag{29}$$

At a height where the horizontal phase speed matches the Alfven speed the ray trajectories for the slow and fast modes possess a common point (which arises from the fold in the wave normal surfaces) at which a mode conversion takes place. For example, in figure 9(*a*), at this point the downward-propagating slow mode is converted to the fast mode which is captured at its critical level, whereas the upward-propagating fast mode is converted to the slow mode which approaches its critical level from below. We note in passing that within the framework of geometric optics (WKBJ approximation) it is not possible to predict whether total reflexion or transmission occurs at the height where the Alfvén speed matches the horizontal phase speed. However, without going into a full wave calculation, there are two reasons for predicting mode conversion at this point. The first reason favouring total transmission is that the energy flux is continuous at this point (Acheson 1972). The other is that the sense of polarization of the wave is preserved by mode conversion at this point.

The ray trajectories for $\alpha = 0$, drawn as broken lines in figure 9, show that the valve effect ceases in this special case essentially because the point at which mode conversion takes place (i.e. where the Alfvén speed equals the horizontal phase speed) moves to $x = +\infty$ in case (*a*) and $x = -\infty$ in case (*b*).

6. Discussion

The application of the condition for the existence of a critical level to the various hydromagnetic waves discussed in this paper implies that critical levels for such waves can exist only if the properties of the medium through which the ray propagates vary in a direction perpendicular to the ambient magnetic field. In this sense critical-level behaviour is exceptional.

We have also shown that critical levels for hydromagnetic waves of the inertial and gravity type exhibit 'valve-like' behaviour, provided that $\mathbf{g}$ (or $\mathbf{\Omega}$) is neither aligned with nor perpendicular to the magnetic field.

The analyses of Acheson (1972) for hydromagnetic-inertial waves and Rudraiah & Venkatachalappa (1972) for a special case of hydromagnetic–gravity–inertial waves, both of which follow Booker & Bretherton (1967), show that the waves are in fact transmitted across a critical level but are heavily attenuated in doing so. On the basis of the boundary-layer-type analysis for gravity waves performed by Hazel (1967), who obtained the same attenuation factor as Booker & Bretherton (1967), it seems reasonable, as Acheson points out, to expect that dissipation is not very important in determining the overall properties of critical levels.

It is not yet established whether hydromagnetic critical levels do or do not exhibit the strong dynamic interaction with the back-ground state that gravity waves do by transferring their energy and momentum to the mean flow near their critical levels. In contrast to gravity waves in a shear flow, hydromagnetic waves still propagate relative to the background near critical levels at which they exhibit a dramatic increase in wave energy density similar to that for gravity waves.

We also wish to emphasize that in some sense critical-level behaviour is a symptom of the WKBJ approximation. The other asymptotic limit in which the mathematics is tractable, i.e. when the vertical wavelength is much larger than the scale length of the variation in properties of the medium, gives different results. For example a study of the reflexion and refraction of gravity waves at a sharp wind shear (McKenzie 1972) shows that those waves that would have been absorbed at a critical level in the WKBJ approximation are in fact reflected and refracted at the wind shear, from which they now extract energy and momentum. A similar study of magnetoacoustic waves incident upon a current-vortex sheet (McKenzie 1970) also exhibits the phenomenon of wave amplification. Similarly, hydromagnetic waves of the gravity or inertial type incident upon a sharp gradient of the magnetic field (a current sheet) will be simply reflected and refracted in a manner determined, essentially, by Snell's law and the boundary conditions appropriate to the discontinuity. In this latter case, however, the phenomenon of wave amplification will not arise if there is no relative streaming motion (shear flow) which can give rise on either side of the discontinuity to the existence of positive and negative energy waves which interact in the presence of the shear to produce wave amplification. Although the phenomena of wave amplification and instability are both related to the presence of streaming motion it is important to recognize that they are quite distinct since the criteria for their occurrence are different.

The author wishes to thank Eric Jamin for helpful discussions and Jan De Leeuw for performing the numerical computations for figures 2 and 7.

REFERENCES

ACHESON, D. J. 1972 The critical level for hydromagnetic waves in a rotating fluid. *J. Fluid Mech.* **53**, 401.

BOOKER, J. R. & BRETHERTON, F. P. 1967 The critical layer for internal gravity waves in a shear flow. *J. Fluid Mech.* **27**, 513.

BRETHERTON, F. P. 1966 The propagation of groups of internal gravity waves in a shear flow. *Quart. J. Roy. Met. Soc.* **92**, 466.

HAZEL, P. 1967 The effect of viscosity and heat conduction on internal gravity waves at a critical level. *J. Fluid Mech.* **30**, 755.

HIDE, R. 1966 Free hydromagnetic oscillations of the Earth's core and the theory of the geomagnetic secular variation. *Phil. Trans. Roy. Soc.* A **259**, 615.

HIDE, R. 1969 On hydromagnetic waves in a stratified rotating incompressible fluid. *J. Fluid Mech.* **39**, 283.

HIDE, R. & STEWARTSON, K. 1972 Hydromagnetic oscillations of the Earth's core. *Rev. Geophys. Space Phys.* **10**, 579.

JONES, W. L. 1968 Reflexion and stability of waves in stably stratified fluids with shear flow: a numerical study. *J. Fluid Mech.* **34**, 609.

LEHNERT, B. 1954 Magnetohydrodynamic waves under the action of the Coriolis force I. *Astrophys. J.* **119**, 647.

LIGHTHILL, M. J. 1960 Studies on magnetohydrodynamic waves and other anisotropic wave motions. *Phil. Trans. Roy. Soc.* A **252**, 397.

LIGHTHILL, M. J. 1965 *J. Inst. Math. Applic.* **1**, 1–28.

LIGHTHILL, M. J. 1967 Predictions on the velocity field coming from acoustic noise and a generalized turbulence in a layer overlaying a convectively unstable atmosphere region. *I.A.U. Symp.* no. 28, p. 429.

MCKENZIE, J. F. 1970 Hydromagnetic wave interaction with the magnetopause and the bow shock. *Planet. Space Sci.* **18**, 1.

MCKENZIE, J. F. 1972 The reflection and amplification of acoustic-gravity waves at a density and velocity discontinuity. *J. Geophys. Res.* **77**, 2915.

RUDRAIAH, N. & VENKATACHALAPPA, M. 1972 Propagation of internal gravity waves in perfectly conducting fluids with shear flow, rotation and transverse magnetic field. *J. Fluid Mech.* **52**, 193.